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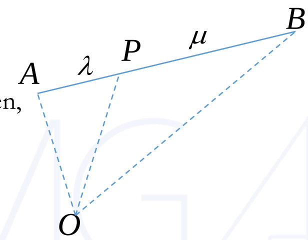
VECTORS

RATIO THEOREM

If point P divides AB in the ratio of $\lambda : \mu$, then

$$\overrightarrow{OP} = \frac{\mu\overrightarrow{OA} + \lambda\overrightarrow{OB}}{\mu + \lambda}$$



Note: If P is the mid-point then,

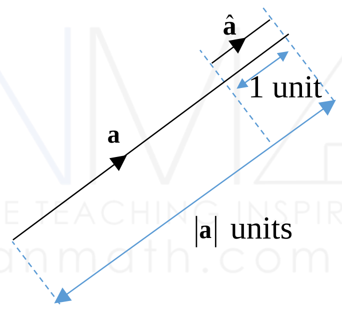
$$\overrightarrow{OP} = \frac{\overrightarrow{OA} + \overrightarrow{OB}}{2}$$

Note: Use Ratio Theorem (Midpoint) for mirror image.

UNIT VECTOR

A vector which has a magnitude (length) of 1. $\Rightarrow |\hat{\mathbf{a}}| = 1$

Unit vector of $\mathbf{a}$, $\hat{\mathbf{a}} = \dfrac{\mathbf{a}}{|\mathbf{a}|}$



SCALAR PRODUCT (DOT PRODUCT)

$$\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta$$

where θ is angle between $\mathbf{a}$ and $\mathbf{b}$.

$$\mathbf{a}\cdot\mathbf{b} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \cdot \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = a_1b_1 + a_2b_2 + a_3b_3$$

E.g. $\begin{pmatrix} -2 \\ 1 \\ 4 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 5 \\ 0 \end{pmatrix} = -4 + 5 + 0 = 1$

SCALAR PRODUCT SPECIAL RESULTS

(i) $\mathbf{a}\cdot\mathbf{a} = |\mathbf{a}|^2$

Since $\mathbf{a} \parallel \mathbf{a} \Rightarrow \theta = 0° \Rightarrow \cos 0° = 1$

(ii) $\mathbf{a}\cdot\mathbf{b} = 0$ when $\mathbf{a} \perp \mathbf{b}$

Since $\theta = 90° \Rightarrow \cos 90° = 0$.

(iii) $\mathbf{a}\cdot\mathbf{b} = \mathbf{b}\cdot\mathbf{a}$

VECTOR PRODUCT (CROSS PRODUCT)

$$\mathbf{a}\times\mathbf{b} = |\mathbf{a}||\mathbf{b}|\sin\theta\ \hat{\mathbf{n}}$$

$$|\mathbf{a}\times\mathbf{b}| = |\mathbf{a}||\mathbf{b}|\sin\theta$$

where $\hat{\mathbf{n}}$ is unit vector perpendicular to both $\mathbf{a}$ and $\mathbf{b}$.

$$\mathbf{a}\times\mathbf{b} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \times \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} a_2b_3 - a_3b_2 \\ -(a_1b_3 - a_3b_1) \\ a_1b_2 - a_2b_1 \end{pmatrix}$$

E.g. $\begin{pmatrix} -2 \\ 1 \\ 4 \end{pmatrix} \times \begin{pmatrix} 2 \\ 5 \\ 0 \end{pmatrix} = \begin{pmatrix} 1\times 0 - 4\times 5 \\ -(-2\times 0 - 4\times 2) \\ -2\times 5 - 1\times 2 \end{pmatrix} = \begin{pmatrix} -20 \\ 8 \\ -12 \end{pmatrix}$

VECTOR PRODUCT SPECIAL RESULTS

(i) $\mathbf{a}\times\mathbf{a} = \mathbf{0}$

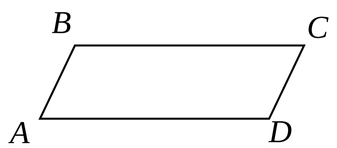
since $\mathbf{a} \parallel \mathbf{a} \Rightarrow \theta = 0° \Rightarrow \sin 0° = 0$

(ii) $\mathbf{a}\times\mathbf{b} = -\mathbf{b}\times\mathbf{a}$

AREA OF PARALLELOGRAM AND TRIANGLE

Area of $ABCD$ $= \left|\overrightarrow{AD}\times\overrightarrow{AB}\right|$

Area of ΔADB $= \dfrac{1}{2}\left|\overrightarrow{AD}\times\overrightarrow{AB}\right|$



PARALLEL VECTORS

If $\mathbf{a} \parallel \mathbf{b}$, then $\mathbf{a} = k\mathbf{b}$, where $k \subseteq \mathbb{R}$

- $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}|$
- $\mathbf{a} \times \mathbf{b} = \mathbf{0}$

PERPENDICULAR VECTORS

If $\mathbf{a} \perp \mathbf{b}$,

- $\mathbf{a} \cdot \mathbf{b} = 0$
- $|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}|$

COLLINEAR POINTS

If points A, B and C are collinear then $\overrightarrow{AB} = k\overrightarrow{BC}$, where $k \subseteq \mathbb{R}$ and B is a common point.

BASIC RESULTS - UNIT VECTORS

Find the unit vector of $3\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$.

Answers: $\frac{1}{\sqrt{29}}(3\mathrm{i} + 4\mathrm{j} - 2\mathrm{k})$

Find the unit vector of $\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$.

Answers: $\frac{1}{\sqrt{21}}(\mathrm{i} + 4\mathrm{j} - 2\mathrm{k})$

Find the possible values of a, given that $(a-1)\mathbf{i} + (1-2a)\mathbf{j} - \frac{1}{2}\mathbf{k}$ is a unit vector.

Answers: $a = \frac{3}{5} \pm \frac{\sqrt{11}}{10}$

APPLICATION OF UNIT VECTOR

The position vectors of two points A and B are $(3\mathbf{i} - \mathbf{j} + 4\mathbf{k})$ and $(2\mathbf{i} + \mathbf{j} + 2\mathbf{k})$ respectively. Points A, B and C are collinear. Given that point C is 6 units away from A, find the possible position vectors of C.

Answers: $\begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix}$ or $\begin{pmatrix} 5 \\ -5 \\ 8 \end{pmatrix}$

The position vectors of two points A and B are $(\mathbf{i} + 3\mathbf{j} + 3\mathbf{k})$ and $(-2\mathbf{i} + 7\mathbf{j} + 3\mathbf{k})$ respectively. Given that point C is on line consisting A and B. And also, C is 15 units away from A, find the possible position vectors of C.

Answers: $\begin{pmatrix} -8 \\ 15 \\ 3 \end{pmatrix}$ or $\begin{pmatrix} 10 \\ -9 \\ 3 \end{pmatrix}$

The position vectors of two points A and B are $(\mathbf{i} - 3\mathbf{j} + 2\mathbf{k})$ and $(4\mathbf{i} + 2\mathbf{k})$ respectively. Given that point C is on line AB and 9 units away from B, find the possible position vectors of C.

Answer: $\begin{pmatrix} 4 + 9\sqrt{2}/2 \\ 9\sqrt{2}/2 \\ 2 \end{pmatrix}$ or $\begin{pmatrix} 4 - 9\sqrt{2}/2 \\ -9\sqrt{2}/2 \\ 2 \end{pmatrix}$

Basic Results - Ratio Theorem

With respect to the origin O , the position vectors of two points A and B are $(6\mathbf{i}-2\mathbf{j}-\mathbf{k})$ and $(-\mathbf{i}+5\mathbf{j}-3\mathbf{k})$ respectively. C is on AB produced such that $2AC=3AB$. Find the position vector of C.

Answer: $-4.5\mathbf{i}+8.5\mathbf{j}-4\mathbf{k}$

Basic Results - Dot Product

The angle between the unit vectors $\mathbf{a}$ and $\mathbf{b}$ is θ.
By expanding the scalar product, show that $(4\mathbf{a}-\mathbf{b})\cdot(\mathbf{a}+4\mathbf{b})=15\cos\theta$.

The angle between the unit vectors $\mathbf{a}$ and $\mathbf{b}$ is θ.
By expanding the scalar product, find $(3\mathbf{a}-\mathbf{b})\cdot(2\mathbf{a}+\mathbf{b})$ in terms of θ.

Answer: $5+\cos\theta$

Basic Results - Magnitude of Vectors

(i) Given two vectors $\mathbf{a}$ and $\mathbf{b}$ such that $|\mathbf{a}| = 5$ and $|\mathbf{b}| = 4$, and that the angle between $\mathbf{a}$ and $\mathbf{b}$ is $\theta = 30°$, find the value of $\mathbf{a} \cdot \mathbf{b}$ in the exact form.

(ii) Another vector $\mathbf{c}$ is such that $\mathbf{c} = 3\mathbf{a} - 2\mathbf{b}$. Find the exact magnitude of $\mathbf{c}$.

Answers: (i) $10\sqrt{3}$ (ii) $\sqrt{289-120\sqrt{3}}$

(i) Given $\mathbf{a}$ is a unit vector and $\mathbf{b}$ is a vector such that $|\mathbf{b}| = 3$, and that the angle between $\mathbf{a}$ and $\mathbf{b}$ is $\theta = 30°$, find the value of $\mathbf{a} \cdot \mathbf{b}$ in the exact form.

(ii) Another vector $\mathbf{c}$ is such that $\mathbf{c} = \mathbf{a} - \mathbf{b}$. Find the exact magnitude of $\mathbf{c}$.

Answers: (i) $\frac{3}{2}\sqrt{3}$ (ii) $\sqrt{10-3\sqrt{3}}$

Basic Results - Application of Dot Product: Perpendicular Vectors

Two unit vectors $\mathbf{a}$ and $\mathbf{b}$ are such that the angle between $\mathbf{a}$ and $\mathbf{b}$ is $60°$.
Given that $\mathbf{x}=2\mathbf{a}-\mathbf{b}$, $\mathbf{y}=-\mathbf{a}+3\mathbf{b}$ and $\mathbf{z}=3\mathbf{a}+\mathbf{b}$.

(i) Show that x is perpendicular to $\mathbf{z}+3\mathbf{y}$.

(ii) Find the magnitude of $\mathbf{x}+2\mathbf{y}+2\mathbf{z}$.

Answer: (ii) $\sqrt{127}$

2012 HCI P2 Q4 (a)

Referred to the origin O, the position vectors of three point A, B and P are $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{a}+5\mathbf{b}$ respectively. Given that $\mathbf{b}$ is a unit vector, the angle AOB is $60°$ and AB is perpendicular to OP, find exact value of $|\mathbf{a}|$. [4]

Answer: $\sqrt{6}-1$

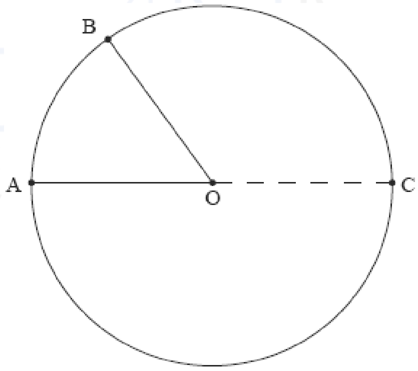
2013 SRJC P2 Q3 (a)(i)

Relative to the origin O, the position vectors of A, B and C are $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ respectively where $\mathbf{c}=3\mathbf{a}-2\mathbf{b}$.

Given that $\mathbf{a}$ is unit vector, $|\mathbf{b}|=3$ and the angle between $\mathbf{a}$ and $\mathbf{b}$ is $\frac{2}{3}\pi$, find the exact value of λ such that $\mathbf{a}$ is perpendicular to $3\mathbf{b}-4\mathbf{a}+\lambda\mathbf{c}$, [3]

Answer: $\lambda=\frac{17}{12}$

The diagram below shows a circle with centre O. The points A, B, C lie on the circumference of the circle and AC is a diameter.



Let $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$

(a) Write down expressions for $\overrightarrow{AB}$ and $\overrightarrow{CB}$ in terms of the vectors $\mathbf{a}$ and $\mathbf{b}$.

(b) Hence prove that angle ABC is a right angle.

Answers: **(a)** $\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$, $\overrightarrow{CB} = \mathbf{a} + \mathbf{b}$

Basic Results - Application of Cross Product: Area of Parallelogram and Triangle

Relative to the origin O, the position vectors of A, B, and C are $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ respectively where $\mathbf{c} = 2\mathbf{a} - 3\mathbf{b}$.

Given that $\mathbf{a}$ is unit vector, $|\mathbf{b}| = 5$ and the angle between $\mathbf{a}$ and $\mathbf{b}$ is $\frac{2}{3}\pi$, show that the exact area of triangle ABC is $\frac{5\sqrt{3}}{2}$ units².

2013 SRJC P2 Q3 (a)(ii)

Relative to the origin O, the position vectors of A, B, and C are $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ respectively where $\mathbf{c} = 3\mathbf{a} - 2\mathbf{b}$.

Given that $\mathbf{a}$ is unit vector, $|\mathbf{b}| = 3$ and the angle between $\mathbf{a}$ and $\mathbf{b}$ is $\frac{2}{3}\pi$, show that the exact area of triangle OBC is $\frac{9\sqrt{3}}{4}$ units^2.

2013 ACJC P2 Q1 (ii)(iii)

Referred to the origin O, the points A and B are such that $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$, where $|\mathbf{a}| = 2$ and $\mathbf{b}$ is a unit vector. The midpoint of OA is M, and the point L on AB is such that $AL : LB = 1 : 2$.

(ii) Find $\overrightarrow{OL}$ in terms of $\mathbf{a}$ and $\mathbf{b}$. Hence find the area of triangle OAL in terms of $\mathbf{a}$ and $\mathbf{b}$.

(iii) Given that LM is perpendicular to AB, show that

$$\mathbf{a} \cdot \mathbf{b} = k$$

where k is a constant to be determined.

Answers: **(ii)** $\overrightarrow{OL} = \frac{2\mathbf{a}+\mathbf{b}}{3}$, $\frac{1}{6}|\mathbf{a} \times \mathbf{b}|$ **(iii)** $k = -2$

2011 IJC P2 Q4

Relative to the origin O, the position vectors of two points A and B are $\mathbf{a}$ and $\mathbf{b}$ respectively. The vector $\mathbf{a}$ is a unit vector which is perpendicular to $\mathbf{a} + 3\mathbf{b}$. The angle between $\mathbf{a}$ and $\mathbf{b}$ is $\frac{2\pi}{3}$.

(i) Show that $|\mathbf{b}| = \frac{2}{3}$. [4]

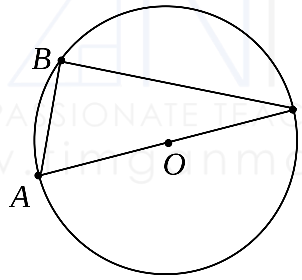
(ii) By expanding $(\mathbf{b} - 2\mathbf{a}) \cdot (\mathbf{b} - 2\mathbf{a})$, find the exact value of $|\mathbf{b} - 2\mathbf{a}|$. [4]

(iii) The point P divides the line AB in the ratio $\lambda : 1 - \lambda$. Find the area of triangle OAP in terms of λ. [5]

Answers: **(ii)** $\frac{2\sqrt{13}}{3}$ **(iii)** $\frac{\lambda\sqrt{3}}{6}$

PRACTICE 1 - BASIC RESULTS

2014 AJC P1 Q2



The points A, B and C lie on a circle with center O and diameter AC. It is given that $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$.

(i) Find $\overrightarrow{BC}$ in terms of $\mathbf{a}$ and $\mathbf{b}$. Hence show that AB is perpendicular to BC. [4]

(ii) Show that the area of triangle ABC can be written as $k|\mathbf{a}\times\mathbf{b}|$ where k is a constant to be found. Hence find, in terms of $|\mathbf{a}|$, the maximum area of triangle ABC. [4]

(iii) Given that $\angle AOB = 30°$, find $\overrightarrow{OF}$ where F is the foot of perpendicular of B to AC. Hence, find $\overrightarrow{OB'}$ where B' is the reflection of B along the line AC. [3]

Answers: **(i)** $\overrightarrow{BC} = -\mathbf{a}-\mathbf{b}$ **(ii)** $|\mathbf{a}\times\mathbf{b}|$, $|\mathbf{a}|^2$ **(iii)** $\overrightarrow{OF} = \frac{\sqrt{3}}{2}\mathbf{a}$, $\overrightarrow{OB'} = \sqrt{3}\mathbf{a}-\mathbf{b}$

2013 RVHS P1 Q9

Referred to an origin, O, the position vectors of two points A and B are $\mathbf{a}$ and $\mathbf{b}$ respectively. $\mathbf{a}$ and $\mathbf{b}$ are not parallel.

(i) The point C lies on AB produced such that $AB : BC$ is $1:3$. Find the position vector of C. [2]

(ii) The point D lies on OB produced such that $OB : OD$ is $1:k$. Given also that CD is perpendicular to AB, show that

$$k = \frac{4|\mathbf{b}|^2 - 7\mathbf{a}\cdot\mathbf{b} + 3|\mathbf{a}|^2}{|\mathbf{b}|^2 - \mathbf{a}\cdot\mathbf{b}}.$$ [3]

(iii) Show that $OADC$ cannot be a parallelogram. [2]

(iv) Find the area of triangle ABD in terms of k. [3]

Answers: **(i)** $4\mathbf{b}-3\mathbf{a}$ **(iv)** $\frac{k-1}{2}|\mathbf{a}\times\mathbf{b}|$

2014 HCI P2 Q3(A)

Referred to the origin O, the points A and B are such that $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$. The point C on OA is such that $OC : CA = 1:2$, the point D on OB is such that $OD : DB = 3:4$ and the point M on CD is such that $11CM = 5CD$.

(i) Find $\overrightarrow{OM}$ in terms of $\mathbf{a}$ and $\mathbf{b}$. [3]

(ii) By considering cross product, find the ratio of the area of triangle OCM to the area of triangle OAB. [3]

Answers: **(a)(i)** $\frac{2}{11}\mathbf{a}+\frac{15}{77}\mathbf{b}$ **(ii)** $5:77$

2015 IJC P1 Q2

Referred to the origin O, the points A and B are such that $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$. The point M is on AB produced such that $AM : BM = 4:1$ and the point N is on OB such that $ON : NB = 2:3$.

(i) Find $\overrightarrow{OM}$ in terms of $\mathbf{a}$ and $\mathbf{b}$.

(ii) By considering cross product, find the ratio of the area of triangle ANB to the area of triangle OAM.

Answers: **(i)** $\dfrac{4\mathbf{b} - \mathbf{a}}{3}$ **(ii)** $9:20$

2015 CJC P1 Q2

With reference to the origin O, two points A and B have position vectors $\mathbf{a}$ and $\mathbf{b}$ respectively. The points O, A and B are not collinear. The point P divides AB in the ratio $AP : PB = 3:2$. It is given that $\mathbf{a}$ is a unit vector, $OB = 4$, angle $AOB = \dfrac{\pi}{3}$ and the foot of the perpendicular from P to the line passing through points O and A is F. Show that $\overrightarrow{OF} = \lambda \mathbf{a}$, where λ is a constant to be determined. [5]

Answers: $\overrightarrow{OF} = \dfrac{8}{5}\mathbf{a}$

2010 NJC P1 Q3

The points A and B have position vectors $\mathbf{a}$ and $\mathbf{b}$ respectively, relative to the origin O such that $|\mathbf{a}| = |\mathbf{b}|$. The point P with position vector $\mathbf{p}$ lies on AB such that $\mathbf{b} \cdot \mathbf{p} = \mathbf{a} \cdot \mathbf{p}$.

(i) Show that AB is perpendicular to OP. [2]

(ii) Determine the position vector of the point D in terms of $\mathbf{a}$ and $\mathbf{b}$, where D is the reflection of O about the line AB. [2]

(iii) Give the geometrical meaning of $|\mathbf{a} \times \mathbf{b}|$. [1]

Answers: **(ii)** $\overrightarrow{OD} = \underset{\sim}{a} + \underset{\sim}{b}$ **(iii)** the area of **rhombus** $OADB$

2008 IJC P1 Q2

In triangle OAB, $\angle OAB = 90°$ and the point C on AB is such that $AC = \dfrac{2}{3}CB$.

With respect to the origin O, the position vectors of A and B are given as $\mathbf{a}$ and $\mathbf{b}$ respectively.

(i) Show that $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}|^2$. [1]

(ii) Find $\mathbf{c}$, the position vector of C in terms of $\mathbf{a}$ and $\mathbf{b}$. [1]

(iii) Given that the lengths of OA and OB are 3 and 5 units respectively, find the length of projection of $\mathbf{c}$ onto $\mathbf{b}$. [3]

Answers: **(ii)** $\mathbf{c} = \dfrac{3}{5}\mathbf{a} + \dfrac{2}{5}\mathbf{b}$ **(iii)** $3\dfrac{2}{25}$